CHAPTER 13

ASYMPTOTIC BEHAVIOR OF NONLINEAR SYSTEMS; STABILITY

1. Asymptotic Stability

The treatment of nonlinear systems presented here will be restricted to local behavior, that is, to the behavior of solutions starting near a known solution of a system.

A solution ψ of a system

$$x' = F(t,x) \qquad \left(' = \frac{d}{dt}\right)$$

which is defined for $t \geqq 0$ is said to be *stable* if, given any $\epsilon > 0$, there exists a $\delta > 0$ such that any solution φ of the system satisfying

$$|\varphi(0) - \psi(0)| < \delta$$

satisfies

$$|\varphi(t) - \psi(t)| < \epsilon \qquad (t \geqq 0)$$

Note that this requires solutions starting nearby $\psi(0)$ to exist for all $t \geqq 0$. The solution ψ is said to be *asymptotically stable* if, in addition to being stable,

$$|\varphi(t) - \psi(t)| \to 0 \qquad (t \to \infty)$$

The following result of Perron is the simplest example of asymptotic stability.

Theorem 1.1. *Let*

$$x' = Ax + f(t,x) \tag{1.1}$$

where A is a real constant matrix with the characteristic roots all having negative real parts. Let f be real, continuous for small $|x|$ and $t \geqq 0$, and

$$f(t,x) = o(|x|) \qquad (|x| \to 0)$$

uniformly in t, $t \geqq 0$. Then the identically zero solution is asymptotically stable.

The conditions that A and f be real or that f be continuous can be replaced by any other conditions which assure the local existence of a solution for (1.1) for small $|x|$ and $t \geqq 0$.

The fact that the characteristic roots of A have negative real parts assures that the linear system $y' = Ay$ has the trivial solution as an asymptotically stable solution.

Proof of Theorem 1.1. The solution φ of (1.1) with $|\varphi(0)|$ small can be continued for increasing t so long as $|\varphi(t)|$ remains small. So long as $\varphi(t)$ exists, it follows from (1.1) that

$$\varphi(t) = e^{tA}\varphi(0) + \int_0^t e^{(t-s)A}f(s,\varphi(s))\,ds \tag{1.2}$$

Because the real parts of the characteristic roots of A are negative, there exist positive constants K and σ such that

$$|e^{tA}| \leqq Ke^{-\sigma t} \qquad (t \geqq 0) \tag{1.3}$$

Using (1.3), (1.2) yields

$$|\varphi(t)| \leqq K|\varphi(0)|e^{-\sigma t} + K\int_0^t e^{-\sigma(t-s)}\,|f(s,\varphi(s))|\,ds$$

Given $\epsilon > 0$, there exists a δ such that $|f(t,x)| \leqq \epsilon|x|/K$ for $|x| \leqq \delta$. Thus, so long as $|\varphi(t)| \leqq \delta$, it follows that

$$e^{\sigma t}|\varphi(t)| \leqq K|\varphi(0)| + \epsilon\int_0^t e^{\sigma s}|\varphi(s)|\,ds$$

This inequality yields, by Prob. 1, Chap. 1,

$$e^{\sigma t}|\varphi(t)| \leqq K|\varphi(0)|e^{\epsilon t}$$

or

$$|\varphi(t)| \leqq K|\varphi(0)|e^{-(\sigma-\epsilon)t} \qquad (t \geqq 0) \tag{1.4}$$

If ϵ is chosen so that $\epsilon < \sigma$, then (1.4) shows that $|\varphi(t)| \leqq K|\varphi(0)|$ so long as $|\varphi(t)| \leqq \delta$. Thus, if $|\varphi(0)| < \delta/K$, it follows that (1.4) is valid for all $t \geqq 0$, which completes the proof of Theorem 1.1.

Let the characteristic roots of A be λ_k, $k = 1, 2, \ldots, n$, and let

$$\max\,(\Re\lambda_k) = -\mu < 0 \tag{1.5}$$

Then *any solution* φ *of* (1.1) *which tends to zero as* $t \to \infty$ *satisfies*

$$\limsup_{t\to\infty} \frac{\log|\varphi(t)|}{t} \leqq -\mu \tag{1.6}$$

Thus, by Theorem 1.1, all solutions with $|\varphi(0)|$ sufficiently small satisfy (1.6).

To prove (1.6), it is noted that σ in (1.3) can be taken as $\mu - \epsilon$ for any given $\epsilon > 0$. This may necessitate taking $K = K_\epsilon$ large. Since $\varphi(t) \to 0$, it is the case that $|\varphi(t_0)|$ can be made as small as is required by taking t_0 large enough. Thus, applying Theorem 1.1 for $t \geqq t_0$, it follows as in (1.4) that

$$e^{(\sigma-\epsilon)(t-t_0)}|\varphi(t)| = 0(1) \qquad (t \to \infty)$$

Since $\sigma = \mu - \epsilon$, it follows that

$$\limsup_{t\to\infty} \frac{\log |\varphi(t)|}{t} \leqq -\mu + 2\epsilon$$

Since $\epsilon > 0$ is arbitrary, (1.6) follows.

A more general statement of Theorem 1.1 weakens the requirement $|f| = o(|x|)$. It is sufficient to assume that, for some $k > 0$,

$$|f(t,x)| \leqq k|x| \qquad (t \geqq 0) \tag{1.7}$$

for all small $|x|$, and that, given any $\epsilon > 0$, there exist δ and T such that

$$|f(t,x)| \leqq \epsilon|x| \qquad (|x| \leqq \delta,\ t \geqq T) \tag{1.8}$$

To show that (1.7) and (1.8) suffice, observe that, with (1.7), (1.1) yields

$$\|\varphi\|' \leqq (\|A\| + kn^{\frac{1}{2}})\|\varphi\|$$

where $\|\varphi\|$ is the Euclidean length of φ. Here use is made of the fact that $n^{-\frac{1}{2}}|x| \leqq \|x\| \leqq |x|$, where x has n components. Thus, so long as $\|\varphi(t)\|$ is small,

$$\|\varphi(t)\| \leqq \|\varphi(0)\|e^{(\|A\|+kn^{\frac{1}{2}})t}$$

or

$$|\varphi(t)| \leqq n^{\frac{1}{2}}|\varphi(0)|e^{(|A|+kn^{\frac{1}{2}})t} \qquad (t \geqq 0) \tag{1.9}$$

so long as $|\varphi(t)|$ is small. In the same way,

$$|\varphi(0)| \leqq n^{\frac{1}{2}}|\varphi(t)|e^{(|A|+kn^{\frac{1}{2}})t} \qquad (t \geqq 0)$$

Having chosen ϵ, (1.8) is used for $t \geqq T$, and Theorem 1.1 is applied to the interval $t \geqq T$, assuming $|\varphi(T)|$ to be small. But by (1.9) it is the case that $|\varphi(T)|$ is small if $|\varphi(0)|$ is small enough. This proves that (1.7) and (1.8) can replace $|f| = o(|x|)$ in Theorem 1.1. It is also the case that (1.6) is valid here.

The inequality (1.9) and that below (1.9) show that stability over $[0,\infty)$ and $[T,\infty)$ are equivalent.

A special case of some interest where (1.7) and (1.8) hold is the case where $f(t,x)$ in (1.1) is replaced by $B(t)x + g(t,x)$, where the matrix $B(t) \to 0$ as $t \to \infty$ and $|g(t,x)| = o(|x|)$ uniformly in $t \geqq 0$ for small $|x|$. In this case, (1.1) would be written as

$$x' = Ax + B(t)x + g(t,x) \tag{1.10}$$

For Theorem 1.1 to be true for $t \geqq T$, it suffices for (1.8) to hold not for arbitrarily small ϵ but merely for $\epsilon < \sigma/K$, where σ and K are from (1.3). With this less restrictive hypothesis, (1.6) need not hold.

In case the matrix A in (1.1) has one or more characteristic roots with positive real parts, then it is not possible for the solution $\varphi = 0$ to be stable. In this sense Theorem 1.1 and the results following it are the best possible.

Theorem 1.2. *Let at least one characteristic root of A in (1.1) have its real part positive. Let $f(t,x)$ satisfy (1.8). Then the solution $\varphi = 0$ of (1.1) is not stable.*

REMARK: With a slightly more restricted hypothesis the result is a consequence of Theorem 4.1 of this chapter.

Proof of Theorem 1.2. To prove the theorem, a transformation $x = Py$, P a constant matrix, is made, resulting in an equation of the form

$$y' = By + g(t,y) \tag{1.11}$$

where $B = P^{-1}AP$. It will be shown that the zero solution of (1.11) is not stable, and this clearly implies that $\varphi = 0$ is not stable for (1.1). By proper choice of P, the matrix B can be put in the form

$$B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} \tag{1.12}$$

where B_1 is a canonical matrix of k rows and columns with its characteristic roots all having positive real parts, while B_2 is a canonical matrix with characteristic roots all having nonpositive real parts. The characteristic roots are in the main diagonal. Those elements off the main diagonal which are not zero can be assumed to be $\gamma > 0$, where γ can be made as small as any assigned positive quantity by proper choice of P. While y corresponding to real x may be complex, Py will be real. Thus

$$g(t,y) = P^{-1}f(t,Py)$$

is defined.

Let the components of φ be φ_j and let

$$R^2 = \sum_{i=1}^{k} |\varphi_i|^2 \qquad \text{and} \qquad \rho^2 = \sum_{i=k+1}^{n} |\varphi_i|^2$$

Let the real parts of the characteristic roots of B_1 exceed some $\sigma > 0$. Choose $\epsilon < \sigma/10$ and choose η and T so that

$$|g(t,y)| \leqq \epsilon\|y\| \qquad (t \geqq T) \tag{1.13}$$

for $\|y\| \leqq \eta$.

Suppose the zero solution of (1.11) is stable. Thus for η and T as chosen above there exists a $\delta > 0$ such that, if φ is a solution of (1.11) with $\rho(T) + R(T) < \delta$, $\rho(t) + R(t) < \eta$ for $t \geqq T$. Choose such a solution φ with $R(T) = 2\rho(T) > 0$.

With σ defined as above, it follows from the use of (1.11), (1.12), and (1.13) that, for $t \geqq T$

$$\sum_{i=1}^{k} (\varphi_i'\bar{\varphi}_i + \varphi_i\bar{\varphi}_i') = 2RR' \geqq 2\sigma R^2 - 2\gamma R^2 - 2\epsilon(\rho + R)R$$

Or, since γ can be chosen smaller than $\sigma/20$ and since ϵ is chosen less than $\sigma/10$, it follows that

$$R' \geqq \tfrac{1}{2}\sigma R - \epsilon\rho \tag{1.14}$$

In the same way,

$$\rho' \leqq \epsilon(\rho + R) + \frac{\sigma}{20}\rho \tag{1.15}$$

From (1.14) and (1.15) follows

$$(R - \rho)' \geqq \tfrac{1}{4}\sigma(R - \rho)$$

Thus

$$R(t) - \rho(t) \geqq (R(T) - \rho(T))e^{\sigma(t-T)/4}$$

Since $R(T) = 2\rho(T)$, it follows that $R(t) \geqq \rho(T)e^{\sigma(t-T)/4}$. This is impossible, since under the hypotheses of stability $\rho(t) + R(t) < \eta$ for $t \geqq T$, and thus the theorem is proved.

Let $f(t,x)$ consist of a linear term $B(t)x$ and a term that for small $|x|$ is $0(|x|^{1+a})$, $a > 0$. An assumption of this kind about f leads to the possibility that, as a function of t, for fixed x, $f(t,x)$ can grow large as $t \to \infty$ without affecting asymptotic stability. This case is treated in the following theorem.

Theorem 1.3. *In Theorem* 1.1 *let the condition* $|f(t,x)| = o(|x|)$ *be replaced by the conditions that for small* $|x|$ *and all* $t \geqq 0$

$$|f(t,x)| \leqq k|x| + |x|^{1+a}t^b \tag{1.16}$$

where $a > 0$, b, *and* k *are constants, and that, given any* $\epsilon > 0$, *there exist* $\delta > 0$ *and* $T \geqq 0$ *such that for* $|x| \leqq \delta$ *and* $t \geqq T$

$$|f(t,x)| \leqq \epsilon|x| + |x|^{1+a}t^b \tag{1.17}$$

Then the solution $\varphi = 0$ *of* (1.1) *is asymptotically stable.*

Proof. With K and σ determined as in (1.3), choose $\eta < \sigma$. Choose ϵ *in* (1.17) so that $\epsilon K < \frac{1}{2}\eta$. The choice of ϵ determines δ and T. From (1.16) for $|x| < 1$ and $0 \leqq t \leqq T$,

$$|f(t,x)| \leqq (k + T^b)|x| \tag{1.18}$$